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# On survival estimation of Lomax distribution under adaptive progressive type-II censoring

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## Abstract

The main objective of the research described in the article is to study the maximum likelihood (ML) estimation and the Bayesian approach for parameter estimation of the Lomax distribution. Additionally, the study aims to determine the approximate intervals for the parameters and the survival function based on adaptive progressive type-II censored data. The ML estimators of the probability distribution's parameters were calculated using the Newton-Raphson method, while the delta method was utilised to compute the approximate confidence intervals for the survival function. The Bayesian approach was also used to estimate the unknown parameters and survival function. This was achieved through the construction of Bayesian estimators under an informative and non-informative prior based on the squared error loss function (SELF) and approximate credible intervals. The Markov Chain Monte Carlo (MCMC) method was employed for this purpose. A Monte Carlo analysis was conducted to test the efficiency of the proposed method in various situations based on different criteria such as mean-squared error, bias, coverage probability, and expected length-estimated criteria. The results indicate that the Bayesian approach out-performs the likelihood method in estimating the Lomax model parameters. Finally, the study includes an application of these methods to real data.

**Key words:** Lomax distribution, maximum likelihood (ML); bayesian estimation; adaptive progressive type-II censoring scheme; squared error loss function (SELF).

## 1. Introduction

The Lomax distribution is a probability distribution that is widely used in reliability and survival analysis. The distribution is named after K. S. Lomax (1954), who first introduced it in 1954. It is a parametric distribution that is used to model the lifetime of products or systems, and it has several applications in engineering, medical sciences, and social sciences. The Lomax distribution is also known as the Pareto Type II distribution. The PDF of the Lomax distribution with shape parameter $\beta$ and scale parameter $\theta$ is given by

$$f(x;\beta,\theta) = \theta\beta(1+\theta x)^{-(\beta+1)}; x,\beta > 0, \theta > 0 \tag{1}$$

and the corresponding Cumulative Distribution Function (CDF) and Survival Function is given as

$$F(x;\beta,\theta) = 1-(1+\theta x)^{-\beta}; x,\beta,\theta > 0 \tag{2}$$

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$$S(t;\beta,\theta) = (1+\theta t)^{-\beta}; t,\beta,\theta > 0 \tag{3}$$

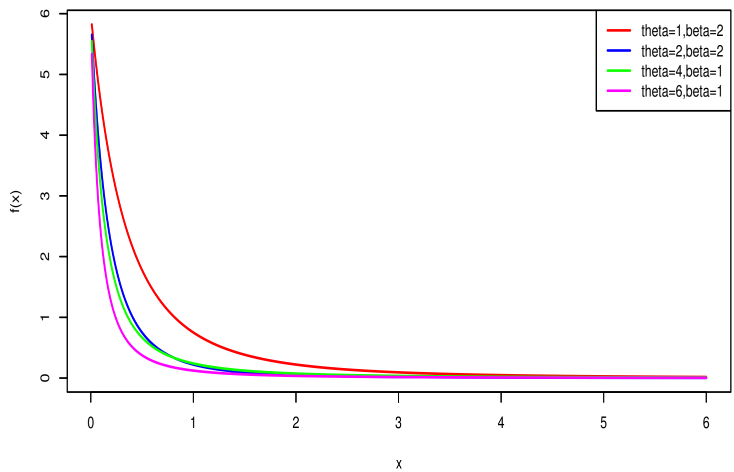


**Figure 1:** PDF of Lomax distribution for different values of parameters

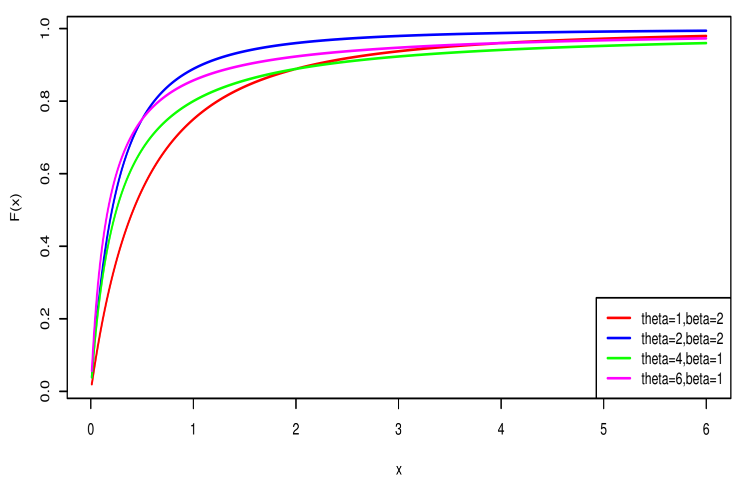


**Figure 2:** CDF of Lomax distribution for different values of parameters

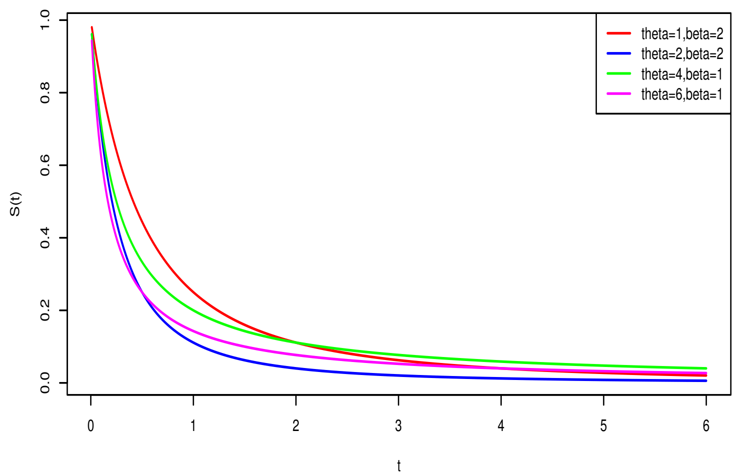


**Figure 3:** Survival Function of Lomax distribution for different values of parameters

The two-parameter Lomax distribution model, traditionally characterized by its shape parameter $\beta$ and scale parameter $\theta$, can be generalized to include the effect of explanatory variables. In this generalized model, the scale parameter $\theta$ is expressed as a function of the covariates through a log-linear relationship. This allows the model to account for the influence of various factors (denoted by Z) on the scale parameter, thereby providing a more flexible and comprehensive framework for modeling data that may be influenced by multiple explanatory variables. When incorporating explanatory variables Z, we often model $\beta$ or $\theta$ (or both) as functions of Z, similar to the approaches used by Altun (2021) and Khan and Khan (2020). These authors demonstrated that using such link functions provides a flexible alternative to models like gamma regression, allowing for more nuanced analysis by accounting for the effects of various control variables on the distribution's parameters. Both MLE and Bayesian Estimation allows for the inclusion of explanatory variables by modeling the parameters of the Lomax distribution as functions of these variables. This generalization enhances the model's applicability in fields like survival analysis and reliability engineering, where understanding the impact of multiple covariates is crucial.

Balkema and de Haan (1974) has used this distribution for reliability and life testing experiment. Hassan and Al-Ghamdi (2009) studied the optimum step stress accelerated life testing for the Lomax distribution using maximum likelihood procedure. In many real-life situations, the lifetime of a product or system is subject to progressive type-II censoring. In such cases, the lifetime of a unit is only observed up to a certain point, and then it is censored. Adaptive progressive type-II censoring is a type of censoring where the sample size changes based on the current state of the experiment. This type of censoring is commonly used in reliability testing and is considered more efficient than traditional censoring methods. In some experiments, it may not be possible to observe the lifetime of all experimental units within the available time. In such cases, censoring is used to reduce the duration and costs associated with the experiment. The two most common censoring schemes are type-I and type-II censoring, which end the experiment at a predetermined time or after a specified number of failures, respectively. However, these schemes lack the flexibility to remove units at points other than the end of the experiment. To address this, a progressive type-II censoring scheme was introduced in real-life tests, and a more flexible scheme called the type-II hybrid progressive censoring was proposed. An adaptive type-II progressive censoring scheme that combines type-I and type-II progressive censoring was also proposed for real-life studies. In a reliability experiment with n identical, independent units, the values of m and n are predetermined and before the experiment begins, a progressive censoring scheme $R = (R_1, ..., Rm)$ is given. It is possible that the experimental total time may exceed the pre-fixed time T. J denotes the observed failure times before the predetermined time T, i.e. $X_{J:m:n} < T < X_{J+1:m:n}, J = 0, 1, ..., m$ where$X_{J:m:n}$, $T < X_{J+1:m:n}, J = 0, 1, ..., m$ where $X_{0:m:n} = 0$ and $X_{m+1:m:n} = \infty$. When the experiment's total time exceeds the ideal test time T, the scheme sets $R_{J+1} = ... = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=J}^{m} R_i$. This allows the experiment to end as soon as possible, with no survival units removed except at the time of the $m^{th}$ failure. There have been several studies on the Lomax distribution under different types of censoring. Cramer and Schmiedt (2011) has considered progressively type-II censored competing risks data from the Lomax distribution and discuss the applicability of the model in the presence of censoring schemes. In recent years, the Adaptive IIPH censoring scheme has been studied by a vast number of authors, including Cui et al. (2019), who discussed the problem of estimating the Weibull distribution parameters in a constant-stress accelerated life test. Sewailem and Baklizi (2019) provided inference for the log-logistic distribution based on an adaptive progressive type-II censoring scheme. Ye et al. (2014) estimated the parameters of the extreme value distribution using the maximum likelihood technique (MLE). Helu and Samawi (2021) studied Statistical analysis based on adaptive progressive hybrid censored data from the Lomax distribution. Helu (2022) discussed Adaptive Type-II Hybrid Progressive Schemes Based on Maximum Product of Spacings for Parameter Estimation of Kumaraswamy Distribution. Nassr et al. (2021) studied statistical inference for the extended Weibull distribution based on adaptive type-II progressive hybrid censored competing risks data. Chen and Gui (2020) discussed the problem of estimating the parameters of the bathtub-shaped failure rate function. Panahi et al. (2021) derived the maximum likelihood and Bayes estimates for the Burr Type-III distribution. Kohansal and Shoaee (2021) studied the statistical inferences for a multicomponent stress-strength reliability model. Okasha et al. (2021) discussed Reliability Estimation of the Lomax Distribution under Adaptive Type-I Progressive Hybrid Censoring Scheme. The purpose of this study is to explore and investigate the Lomax distribution under adaptive progressive type-II censoring. Specifically, this study aims to estimate the parameters and survival function of the Lomax distribution based on the adaptive progressive type-II censored data.

The structure of the article is as follows: Section 1 provides an introduction, outlining the research problem and objectives. Section 2 focuses on estimating the parameters and survival function using Maximum Likelihood Estimation (MLE). Section 3 presents the confidence intervals for the parameters and survival function. Section 4 presents the Bayesian estimators for the parameters and survival function based on SELF. To assess the performance of the estimators, a simulation study is conducted in Section 5 and the estimators are compared using the R software. Section 6 presents the analysis of a real-life dataset to demonstrate the practical application of the proposed estimators. Finally, in Section 7, the article concludes by summarizing the key findings and implications of the study.

## 2. Maximum Likelihood Estimation

Suppose that $X^R_{1:m:n}, X^R_{2:m:n}, \ldots X^R_{m:m:n}$ is an adaptive progressive type-II censored sample of size m from a sample of size n with censoring scheme $R = (R_1, R_2, \ldots, R_m)$ taken from distribution having f(x) as the PDF and F(x) as the CDF, and $X_{J:m:n}$ is the last observed failure before T which is prefixed best testing time. The observed values of an adaptive type-II progressively censored sample are represented by $x = x^R_{1:m:n}, x^R_{2:m:n}, \ldots x^R_{m:m:n}$ (simplified as $x = x_1, x_2, \ldots x_m$ in later equations). On this basis, the corresponding likelihood function is given by

$$L(x^R_{1:m:n}, x^R_{2:m:n}, \ldots, x^R_{m:m:n}) = D_J \prod_{i=1}^{m} f(x_{i:m:n}) \left[\prod_{i=1}^{J} (1 - F(x_{i:m:n}))\right]^{R_i} \left[(1 - F(x_{m:m:n}))\right]^{R_J} \quad (4)$$

$D_J = \prod_{i=1}^{m} [n - i + 1 - \sum_{k=1}^{max(i-1,J)} R_k]$ and $R_J = n - m - \sum_{i=i}^{J} R_i$.

The Likelihood function for $x^R_{1:m:n}, x^R_{2:m:n}, \ldots, x^R_{m:m:n}$based on the Lomax distribution is written as

$$L(\beta, \theta; x) = D_J \prod_{i=1}^{m} \left[\theta\beta(1 + \theta x_i)^{-(\beta+1)}\right] \left[\prod_{i=1}^{J} (1 + \theta x_i)^{-\beta}\right]^{R_i} \left[(1 + \theta x_m)^{-\beta}\right]^{R_J} \quad (5)$$

Further, the log-likelihood function can be written as

$$\ln L(\beta, \theta; x) = m\ln(\theta) + m\ln(\beta) - (\beta + 1)\sum_{i=1}^{m} \ln(1 + \theta x_i) - \beta \sum_{i=1}^{J} R_i \ln(1 + \theta x_i) - \beta R_J \ln(1 + \theta x_m) \quad (6)$$

Then, take the partial derivative of the log-likelihood function, and obtain the likelihood equations as:

$$\frac{\partial \ln L(\beta, \theta; x)}{\partial \theta} = \frac{m}{\theta} - (\beta + 1)\sum_{i=1}^{m} \frac{x_i}{1 + \theta x_i} - \beta \sum_{i=1}^{J} \frac{R_i x_i}{1 + \theta x_i} - \beta R_J \frac{x_m}{1 + \theta x_m} = 0 \quad (7)$$

$$\frac{\partial \ln L(\beta, \theta; x)}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m} \ln(1 + \theta x_i) - \sum_{i=1}^{J} R_i \ln(1 + \theta x_i) - R_J(1 + \theta x_m) = 0 \quad (8)$$

Equations (7) and (8) cannot be solved for $\beta$ and $\theta$ explicitly. So, these equations required numerical solving.

The ML estimator for the survival function by using the invariance property of ML estimator is as follows:

$$\hat{S(t)} = (1+\hat{\theta}t)^{-\hat{\beta}} \tag{9}$$

## 3. Asymptotic Confidence Intervals

The Fisher information matrix was discussed by Aldrich (1997) and the consequently observed Fisher information matrix of the parameters $\beta$ and $\theta$ for large n, is given as follows:

$$I(\hat{\beta},\hat{\theta}) = \begin{bmatrix} -\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\beta^2} & -\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\beta\partial\theta} \\ -\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\theta\partial\beta} & -\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\theta^2} \end{bmatrix}_{\hat{\beta},\hat{\theta}} \tag{10}$$

where

$$\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\beta^2} = -\frac{m}{\beta^2}$$

$$\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\theta^2} = -\frac{m}{\theta^2} + (\beta+1)\sum_{i=1}^{m}\frac{x_i^2}{(1+\theta x_i)^2} + \beta\sum_{i=1}^{J}\frac{R_i x_i^2}{(1+\theta x_i^2)} + \beta R_J \frac{x_m^2}{(1+\theta x_m)^2}$$

$$\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\theta\partial\beta} = -\sum_{i=1}^{m}\frac{x_i}{(1+\theta x_i)} - \sum_{i=1}^{J}\frac{R_i x_i}{(1+\theta x_i)} - R_J\frac{x_m}{(1+\theta x_m)}$$

$$\frac{\partial^2 \ln L(\beta,\theta;x)}{\partial\beta\partial\theta} = -\sum_{i=1}^{m}\frac{x_i}{(1+\theta x_i)} - \sum_{i=1}^{J}\frac{R_i x_i}{(1+\theta x_i)} - R_J\frac{x_m}{(1+\theta x_m)}$$

It is difficult to find the expected Fisher information analytically. Therefore, by using the concept of large sample theory and the variance covariance matrix, which is the inverse of the observed Fisher information matrix $I^{-1}(\hat{\beta},\hat{\theta})$, the approximate $100(1-\alpha)$ normal confidence intervals for the parameters $\beta$ and $\theta$ are given respectively as

$$\left(\hat{\beta} - z_{\frac{\alpha}{2}}\sqrt{var(\hat{\beta})}, \hat{\beta} + z_{\frac{\alpha}{2}}\sqrt{var(\hat{\beta})}\right) \tag{11}$$

$$\left(\hat{\theta} - z_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta})}, \hat{\theta} + z_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta})}\right) \tag{12}$$

where $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution N(0,1) with right-tail probability $\frac{\alpha}{2}$. In addition, the Delta method (Greene, 2010), is applied to evaluate the approximate confidence intervals for the survival functions S(t). This is a natural way for calculating the confidence interval for the functions of the ML estimators, in which these functions are intractable to calculating the variance analytically. Then, we create linear approximations of this survival function and then calculate the variance of linear approximation as follows:

$$C = \begin{pmatrix} \frac{\partial S(t)}{\partial\beta} & \frac{\partial S(t)}{\partial\theta} \end{pmatrix} \tag{13}$$

where

$$\frac{\partial S(t)}{\partial\beta} = -(1+\theta t)^{-\beta}.\ln(1+\theta t) \tag{14}$$

$$\frac{\partial S(t)}{\partial \theta} = -\beta t(1+\theta t)^{(-\beta-1)} \tag{15}$$

The approximate estimate of the variance of S(t) is given by the following:

$$var(\hat{S(t)}) = \left[C^t I^{-1}(\beta,\theta)C\right]_{\hat{\beta},\hat{\theta}}$$

Then, the approximate confidence interval for S(t) is as follows:

$$\left(\hat{S(t)} - z_{\frac{\alpha}{2}}\sqrt{var(\hat{S(t)})}, \hat{S(t)} + z_{\frac{\alpha}{2}}\sqrt{var(\hat{S(t)})}\right) \tag{16}$$

where $z_{\frac{\alpha}{2}}$ is the upper $(\frac{\alpha}{2})^{th}$ quantile of the standardized normal distribution.

## 4. Bayesian Estimation

Bayesian estimation is a statistical method for estimating the parameters of a probability distribution based on prior knowledge and observed data. In this approach, the unknown parameters are treated as random variables with their own prior probability distributions, and the observed data are used to update these prior distributions to obtain a posterior distribution that reflects both the prior information and the new evidence provided by the data. It includes the ability to incorporate prior knowledge into the analysis, the flexibility to handle complex models and data structures, and the ability to quantify uncertainty in a more intuitive way than traditional frequentist methods. In this paper, the Bayes estimates under the Squared Error Loss Function (SELF) are constructed for the unknown parameters $(\theta,\beta)$ and for the survival function. The corresponding credible intervals for these quantities are calculated. It is supposed that the unknown parameters $\beta$ and $\theta$ are independent and follow the gamma distributions as

$$\pi(\beta) \propto \beta^{a_1-1}e^{-b_1\beta}; a_1, b_1 > 0$$

$$\pi(\theta) \propto \theta^{a_2-1}e^{-b_2\theta}; a_2, b_2 > 0$$

Thus, the joint prior distribution becomes

$$\pi(\beta,\theta) \propto \beta^{a_1-1}\theta^{a_2-1}e^{-(b_1\beta+b_2\theta)} \tag{17}$$

The non-informative priors for both parameters $\beta$ and $\theta$ are considered to be $\pi_1(\theta) \propto 1$ and $\pi_2(\beta|\theta) \propto \frac{1}{\beta}$. When $\pi_1(\theta)$ is multiplied by the $\pi_2(\beta|\theta)$, corresponding prior density of $\beta$ and $\theta$ is given by $\pi(\beta,\theta) = \pi_1(\theta) * \pi_2(\beta|\theta)$; Clearly, $\pi(\beta,\theta) \propto \frac{1}{\beta}$. Subsequently, the general form of the posterior density is proportional to the likelihood function time of the prior density function, as follows:

$$p(\beta,\theta|x) \propto (likelihood \times prior)$$

And the corresponding joint posterior conditional density function with informative priors is

$$p(\beta,\theta|x) \propto \left[\prod_{i=1}^{m}\theta\beta(1+\theta x_i)^{-(\beta+1)}\right]\left[\prod_{i=1}^{J}(1+\theta x_i)^{-\beta}\right]^{R_i}\left[(1+\theta x_m)^{-\beta}\right]^{n-m-\sum_{i=i}^{J}R_i} \times\beta^{a_1-1}\theta^{a_2-1}e^{-(b_1\beta+b_2\theta)} \quad (18)$$

The corresponding joint posterior conditional density function with non-informative priors is

$$p(\beta,\theta|x) \propto \left[\prod_{i=1}^{m}\theta\beta(1+\theta x_i)^{-(\beta+1)}\right]\left[\prod_{i=1}^{J}(1+\theta x_i)^{-\beta}\right]^{R_i}\left[(1+\theta x_m)^{-\beta}\right]^{n-m-\sum_{i=i}^{J}R_i} \times \frac{1}{\beta} \quad (19)$$

Hence, the Bayes estimates of any function of $\theta$ and $\beta$ such as $g(\beta,\theta)$, based on SELF is obtained as

$$\widehat{g(\beta,\theta)} = E_{\beta,\theta|x}\widehat{g(\beta,\theta)} = \frac{\int_0^\infty\int_0^\infty \widehat{g(\beta,\theta)}L(\beta,\theta|x)\times\pi(\beta,\theta)d\beta d\theta}{\int_0^\infty\int_0^\infty L(\beta,\theta|x)\times\pi(\beta,\theta)d\beta d\theta} \quad (20)$$

Clearly, calculating the Bayes estimators using (18), (19) and (20) analytically is unattainable. As a result, we advocate employing the MCMC technique to obtain the Bayes estimates of $\theta$ and $\beta$ and the associated credible intervals. The Metropolis-Hastings algorithm is a Markov chain Monte Carlo (MCMC) method for sampling from a probability distribution that is difficult to sample directly. It is a general algorithm that can be used to sample from any distribution, as long as the distribution can be evaluated up to a constant proportionality factor. The algorithm works by defining a proposal distribution, which is used to generate a candidate sample from the current state of the chain. The candidate sample is then accepted or rejected based on the probability of moving from the current state to the candidate state, as determined by a Metropolis-Hastings acceptance probability. To apply the MCMC technique, we should first derive the full conditional distributions of $\beta$ and $\theta$ as follows:

$$h(\beta|\theta,x) \propto \beta^{m+a_1-1}e^{b_1\beta}\prod_{i=1}^{m}\left[(1+\theta x_i)^{-(\beta+1)}\right]\left[\prod_{i=1}^{J}(1+\theta x_i)^{-\beta}\right]^{R_i}\left[(1+\theta x_m)^{-\beta}\right]^{R_J} \quad (21)$$

$$h(\theta|\beta,x) \propto \theta^{m+a_2-1}e^{b_2\theta}\prod_{i=1}^{m}\left[(1+\theta x_i)^{-(\beta+1)}\right]\left[\prod_{i=1}^{J}(1+\theta x_i)^{-\beta}\right]^{R_i}\left[(1+\theta x_m)^{-\beta}\right]^{R_J} \quad (22)$$

To involve the MH sampling, we assume the normal distribution as the proposal distribution to acquire the Bayesian estimates and to obtain the credible intervals. Here, we simulate samples from the full conditional posterior distribution and the proposal proceeds by proposing a joint move on $(\theta, \beta)$. The Metropolis-Hasting algorithm is illustrated below.

1) Initialize j=0, $\theta^{(j)} = 1.5$, $\beta^{(j)} = 1$
2) j=1

3) Generate $\theta$ and $\beta$ using normal candidate distribution.
4) Compute the acceptance probability s= $\min\left(1, \frac{p(\theta^*|data)}{p(\theta^{j-1}|data)} \frac{f(\theta^{j-1}|\theta^*)}{f(\theta^*|\theta^{j-1})}\right)$
5) Draw u from a uniform (0,1) density.
6) If $u \leq r$; set $\theta^j = \theta^*$ and otherwise $\theta^j = \theta^{j-1}$
8) Increment j and repeat steps 3 to 6 for N = 11,000 times.
9) Approximate Bayes estimates of $\theta$ and $\beta$ using MCMC samples based on the SELF as $\hat{\theta}_B = \frac{1}{N-M}\sum_{i=M+1}^{N}\theta^{(i)}$ and $\hat{\beta}_B = \frac{1}{N-M}\sum_{i=M+1}^{N}\beta^{(i)}$ where M is burn-in.
10) An approximate Bayesian estimates of the S(t), based on the SELF, can be found as $\widehat{S(t)}_B = \frac{1}{N-M}\sum_{i=M+1}^{N} S^{(i)}(t)$
11) Compute the credible intervals of $\theta$ and $\beta$, order $\theta_{M+1}, \theta_{M+2}..., \theta_N$ and $\beta_{M+1}, \beta_{M+2}..., \beta_N$ as $\theta_1, \theta_2..., \theta_{N-M}$ and $\beta_M, \beta_M..., \beta_{N-M}$ Then, the $100(1-\alpha)\%$ symmetric credible intervals of $\theta$ and $\beta$ constructed as $\left(\theta_{((N-M)(\frac{\alpha}{2}))}, \theta_{((N-M)(1-\frac{\alpha}{2}))}\right)$ and $\left(\beta_{((N-M)(\frac{\alpha}{2}))}, \beta_{((N-M)(1-\frac{\alpha}{2}))}\right)$.
12) Compute the credible intervals of $S(t)$ order $S_{M+1}(t), S_{M+2}(t)..., S_N(t)$ as $S_1(t) < S_2(t) < ... < S_{N-M}(t)$Then, the $100(1-\alpha)\%$ symmetric credible intervals of $\theta$ and $\beta$ constructed as $\left(S_{((N-M)(\frac{\alpha}{2}))}(t), S_{((N-M)(1-\frac{\alpha}{2}))}(t)\right)$.

## 5. Simulation Study

In this section, Monte Carlo simulations are performed to know the performance of the proposed estimators developed in the previous sections of the parameters, the survival function based on an adaptive progressive type-II censoring scheme. The process of generating an adaptive progressive type-II censored sample with a pre-determined number of n and m and the progressive censoring schemes with given values of the ideal censoring time T from the Lomax distribution is described below using the procedure described by Balakrishnan and Sandhu (1995) and by Ng et al. (2009). The steps are as follows:
1) Define the values of n, m, $\theta$, $\beta$, T and $R = (R_1, R_2...R_m)$.
2) Simulate m random variables from uniform (0,1) as $W_1, W_2...W_m$.
3) Set $V_i = W_i^{\frac{1}{(i+R_m+R_{m-1}+...+R_{m-i+1})}}$ for i=1,2,...m.
4) Set $U_i = V_m V_{m-1}...V_{m-i+1}$, for i=1,2,...m. Then, $U_1, U_2, ...U_m$, is the m progressive type-II observed sample from the Uniform (0,1) distribution.
5) Set $x_i = F^{-1}(Ui)$ for i=1,2,... m, where $F^{-1}(Ui)$represent the quantile function of the Lomax distribution. Thus, $x_1, x_2, ..., x_m$, is the needed progressive type-II observed sample from the specified distribution F(.) by using the inverse transformation method.
6) Identify the value of J, where $x_{J:m:n} < T < x_{J+1:m:n}$, discard the sample $x_{j+2:m:n}, ..., x_{m:m:n}$.
7) Simulate the first m -J -1 order statistics from a truncated distribution considered as $\frac{f(x)}{[1-f(x_{J+1:m:n})]}$ with sample size $\left(n - \sum_{i=1}^{j} R_i - J - 1\right)$ as $x_{j+2:m:n}, x_{j+3:m:n}, ..., x_{m:m:n}$.

Hence, a simulation study was executed using the ideal total test time T=1. To generate the data, we supposed that the initial true values of the parameters $\theta$ and $\beta$ were (1.5, 1), we used the values of t= 0.5, 1, the corresponding values of the survival function are S(t) are 0.5714 and 0.4 respectively. For prior information, the hyperparameters ($a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 1$) were considered. To find the Bayesian estimates and the 95% Bayes intervals for the unknown parameters, we simulate 10,000 MCMC values from the target distribution using the Metropolis–Hastings algorithm.

**Table 1:** Average Estimate(AE), Bias, MSE, AL and CP of scale($\theta$) and shape($\beta$) Parameters Based on T=1

| **(n,m)** | **CS** | **MLE** | | **Bayes Informative** | | **Bayes Non-Informative** | |
|---|---|---|---|---|---|---|---|
| **(50,20)** | $(20,0^{19})$ | $\theta$ | $\beta$ | $\theta$ | $\beta$ | $\theta$ | $\beta$ |
| **AE** | | 1.7038 | 1.1901 | 1.3942 | 0.9590 | 1.3112 | 1.0904 |
| **Bias** | | 0.2038 | 0.1901 | 0.1058 | 0.0410 | 0.1888 | 0.1090 |
| **MSE** | | 1.5171 | 1.5063 | 0.8998 | 0.9327 | 1.527 | 0.7821 |
| **CI** | | (-0.7851,0.9574) | (0.12011,1.2738) | (0.6177,1.7736) | (0.5848,1.6417) | (0.5283,1.6523) | (0.6384,1.7601) |
| **AL** | | 1.7426 | 1.15374 | 1.1559 | 1.0569 | 1.1239 | 1.1216 |
| **CP** | | 0.912 | 0.905 | 0.925 | 0.965 | 0.901 | 0.945 |
| **AE** | $(2^5,1^{10},0^5)$ | 1.5862 | 0.9441 | 1.4156 | 1.1901 | 1.2270 | 1.1964 |
| **Bias** | | 0.0862 | -0.0558 | -0.0843 | 0.1905 | -0.2729 | -0.1964 |
| **MSE** | | 1.4522 | 1.1055 | 0.9428 | 0.7291 | 1.4048 | 0.9706 |
| **CI** | | (-0.9843,0.7409) | (-0.0415,1.2507) | (0.7000,1.7187) | (0.6674,1.6002) | (0.5484,1.6839) | (0.5979,1.7106) |
| **AL** | | 1.7252 | 1.2922 | 1.0186 | 1.0328 | 1.1355 | 1.1127 |
| **CP** | | 0.930 | 0.935 | 0.920 | 0.95 | 0.93 | 0.985 |
| **AE** | $(1^{20})$ | 1.7997 | 0.9693 | 1.3257 | 0.8897 | 1.4754 | 1.1809 |
| **Bias** | | 0.2997 | 0.0306 | -0.1742 | -0.1102 | -0.0245 | 0.1809 |
| **MSE** | | 1.6223 | 1.3605 | 1.0921 | 0.7793 | 0.5040 | 0.3043 |
| **CI** | | (-0.8943,0.9702) | (0.1150,1.2709) | (0.6177,1.7736) | (0.5848,1.6417) | (0.6042,1.7491) | (0.5964,1.7373) |
| **AL** | | 1.8646 | 1.1558 | 1.1216 | 1.0895 | 1.4493 | 1.1409 |
| **CP** | | 0.919 | 0.925 | 0.905 | 0.975 | 0.91 | 0.97 |
| **(70,30)** | $(30,0^{29})$ | | | | | | |
| **AE** | | 1.3252 | 1.0809 | 1.7165 | 1.0109 | 1.7409 | 0.8815 |
| **Bias** | | 0.1748 | -0.0809 | 0.2165 | 0.0109 | 0.2409 | -0.1184 |
| **MSE** | | 1.1773 | 1.1999 | 0.7331 | 0.5108 | 0.9231 | 0.4391 |
| **CI** | | (-0.7000,1.0940) | (-0.0109,1.4870) | (0.5838,1.7535) | (0.6278,1.7433) | (0.5589,1.6820) | (0.5800,1.6676) |
| **AL** | | 1.7241 | 1.1980 | 1.0696 | 1.0154 | 1.1031 | 1.0875 |
| **CP** | | 0.92 | 0.95 | 0.915 | 0.965 | 0.915 | 0.975 |
| **AE** | $(2^{20},1^{10},0^{10})$ | 1.4701 | 1.0816 | 1.4667 | 1.0401 | 1.5763 | 0.9887 |
| **Bias** | | -0.0298 | 0.0816 | -0.0332 | 0.0401 | 0.0763 | -0.0112 |
| **MSE** | | 1.1836 | 1.0439 | 0.0520 | 0.0190 | 0.9508 | 0.3430 |
| **CI** | | (-0.8295,0.9942) | (-0.1112,1.3783) | (0.6065,1.7594) | (0.5945,1.6390) | (0.5847,1.7599) | (0.5972,1.6978) |
| **AL** | | 1.5238 | 1.1896 | 1.0052 | 1.0044 | 1.0752 | 1.1006 |
| **CP** | | 0.915 | 0.945 | 0.910 | 0.945 | 0.91 | 0.97 |
| **AE** | $(1^{30})$ | 1.7147 | 0.9441 | 1.3224 | 1.0036 | 1.4864 | 1.0687 |
| **Bias** | | 0.2147 | -0.0558 | -0.1775 | 0.0036 | -0.0135 | 0.0687 |
| **MSE** | | 1.5038 | 0.9678 | 0.8607 | 0.5595 | 0.2036 | 0.0937 |
| **CI** | | (-0.9605,0.7143) | (-0.0223,1.3828) | (0.6742,1.7820) | (0.6395,1.6097) | (0.6317,1.7392) | (0.6522,1.7104) |
| **AL** | | 1.6749 | 1.2052 | 1.1078 | 0.9002 | 1.1074 | 1.0582 |
| **CP** | | 0.922 | 0.910 | 0.91 | 0.945 | 0.905 | 0.95 |
| **(90,40)** | $(40,0^{39})$ | | | | | | |
| **AE** | | 1.3552 | 1.0911 | 1.4690 | 1.0950 | 1.2791 | 0.8908 |
| **Bias** | | -0.1448 | 0.0911 | -0.0309 | 0.0950 | -0.2208 | 0.1092 |
| **MSE** | | 0.8560 | 0.8938 | 0.6611 | 0.5093 | 0.7338 | 0.3827 |
| **CI** | | (-0.4553,1.3351) | (0.1222,1.8344) | (0.5587,1.6813) | (0.6407,1.7708) | (0.6552,1.7481) | (0.6160,1.5899) |
| **AL** | | 1.6905 | 1.0122 | 1.0226 | 1.0250 | 1.0929 | 1.1138 |
| **CP** | | 0.91 | 0.93 | 0.90 | 0.95 | 0.925 | 0.975 |
| **AE** | $(2^{20},1^{10},0^{10})$ | 1.4701 | 1.2008 | 1.4767 | 1.0201 | 1.3709 | 1.0938 |
| **Bias** | | -0.0298 | 0.2008 | -0.0233 | 0.0201 | -0.1290 | -0.0938 |
| **MSE** | | 0.7395 | 0.8214 | 0.0420 | 0.0160 | 0.4358 | 0.2306 |
| **CI** | | (-0.1921,1.3903) | (0.7361,1.8359) | (0.5741,1.7177) | (0.5033,1.5436) | (0.5534,1.7066) | (0.5505,1.5849) |
| **AL** | | 1.4824 | 0.9370 | 0.9836 | 0.9943 | 1.0031 | 1.0344 |
| **CP** | | 0.905 | 0.925 | 0.90 | 0.94 | 0.93 | 0.955 |
| **AE** | $(1^{40})$ | 1.6753 | 1.0246 | 1.2259 | 1.2099 | 1.6934 | 1.0161 |
| **Bias** | | 0.1752 | 0.0246 | -0.2640 | 0.2099 | 0.1934 | 0.0161 |
| **MSE** | | 0.8927 | 0.7349 | 0.8419 | 0.5322 | 0.1945 | 0.0083 |
| **CI** | | (-0.7775,0.8605) | (-0.0972,1.4627) | (0.6231,1.7030) | (0.6549,1.7312) | (0.6401,1.7281) | (0.6157,1.6686) |
| **AL** | | 1.6381 | 1.0599 | 1.0798 | 1.0062 | 1.0880 | 1.0528 |
| **CP** | | 0.910 | 0.930 | 0.905 | 0.955 | 0.90 | 0.97 |

**Table 2:** Average Estimate(AE), Bias, MSE, AL and CP of S(t), t=0.5, 1 Parameters Based on T=1

| (n,m) | CS | MLE | | Bayes Informative | | Bayes Non-Informative | |
|---|---|---|---|---|---|---|---|
| (50,20) | $(20,0^{19})$ | S(0.5) | S(1) | S(0.5) | S(1) | S(0.5) | S(1) |
| AE | | 0.7856 | 0.5863 | 0.5422 | 0.3675 | 0.6225 | 0.4614 |
| Bias | | 0.2141 | 0.1863 | -0.0292 | -0.0325 | 0.0510 | 0.0614 |
| MSE | | 0.1111 | 0.1259 | 0.0041 | 0.0.0046 | 0.0096 | 0.0141 |
| CI | | (0.6302,0.9456) | (0.4404,0.9110) | (0.5164,0.5685) | (0.3383,0.3977) | (0.5277,0.7128) | (0.3488,0.5709) |
| AL | | 0.3153 | 0.4705 | 0.0578 | 0.0671 | 0.1864 | 0.2221 |
| CP | | 0.925 | 0.915 | 0.943 | 0.952 | 0.925 | 0.935 |
| AE | $(2^5,1^{10},0^5)$ | 0.7694 | 0.5671 | 0.6107 | 0.4471 | 0.5889 | 0.4248 |
| Bias | | 0.1980 | 0.1671 | 0.0392 | 0.0471 | 0.0175 | 0.0248 |
| MSE | | 0.0953 | 0.0959 | 0.0711 | 0.0191 | 0.0252 | 0.0388 |
| CI | | (0.6041,0.9072) | (0.4250,0.8495) | (0.5245,0.7034) | (0.3447,0.5597) | (0.5362,0.6298) | (0.3587,0.4770) |
| AL | | 0.3030 | 0.4245 | 0.1788 | 0.2150 | 0.1936 | 0.2402 |
| CP | | 0.915 | 0.92 | 0.935 | 0.94 | 0.93 | 0.905 |
| AE | $(1^{20})$ | 0.7398 | 0.5418 | 0.6063 | 0.4414 | 0.6169 | 0.4584 |
| Bias | | 0.1684 | 0.1418 | 0.0348 | 0.0413 | 0.0455 | 0.0584 |
| MSE | | 0.1134 | 0.1022 | 0.0091 | 0.0121 | 0.0158 | 0.0238 |
| CI | | (0.5214,0.7963) | (0.4072,0.7662) | (0.5217,0.7053) | (0.3442,0.5647) | (0.5699,0.8347) | (0.3938,0.7460) |
| AL | | 0.2748 | 0.3589 | 0.1836 | 0.2205 | 0.2648 | 0.3481 |
| CP | | 0.92 | 0.93 | 0.945 | 0.95 | 0.925 | 0.93 |
| (70,30) | $(30,0^{29})$ | | | | | | |
| AE | | 0.7514 | 0.5366 | 0.5422 | 0.3675 | 0.6231 | 0.4602 |
| Bias | | 0.1799 | 0.1366 | -0.0292 | -0.0325 | 0.0517 | 0.0602 |
| MSE | | 0.0820 | 0.0696 | 0.0041 | 0.0046 | 0.0094 | 0.0131 |
| CI | | (0.5632,0.8759) | (0.3614,0.7911) | (0.5164,0.5685) | (0.3383,0.3977) | (0.5306,0.7171) | (0.3519,0.5753) |
| AL | | 0.3127 | 0.4297 | 0.1751 | 0.2203 | 0.0521 | 0.0594 |
| CP | | 0.925 | 0.93 | 0.955 | 0.95 | 0.91 | 0.935 |
| AE | $(2^{10},1^{10},0^{10})$ | 0.7391 | 0.5236 | 0.6165 | 0.4537 | 0.6605 | 0.5134 |
| Bias | | 0.1676 | 0.1238 | 0.0451 | 0.0537 | 0.0891 | 0.0113 |
| MSE | | 0.0719 | 0.0537 | 0.0210 | 0.0172 | 0.0119 | 0.0166 |
| CI | | (0.5537,0.8362) | (0.3654,0.7352) | (0.5211,0.7150) | (0.3423,0.5731) | (0.5533,0.7352) | (0.3770,0.6174) |
| AL | | 0.2825 | 0.3697 | 0.1639 | 0.2008 | 0.1818 | 0.2103 |
| CP | | 0.905 | 0.92 | 0.935 | 0.935 | 0.92 | 0.93 |
| AE | $(1^{30})$ | 0.7391 | 0.5284 | 0.6136 | 0.4487 | 0.6824 | 0.5395 |
| Bias | | 0.1667 | 0.1284 | 0.0422 | 0.0487 | 0.1110 | 0.1395 |
| MSE | | 0.0796 | 0.0632 | 0.00811 | 0.0113 | 0.0126 | 0.0125 |
| CI | | (0.5561,0.8299) | (0.3806,0.7390) | (0.5256,0.7040) | (0.3466,0.5611) | (0.5539,0.7685) | (0.3776,0.6608) |
| AL | | 0.2738 | 0.3583 | 0.1784 | 0.2144 | 0.2145 | 0.2832 |
| CP | | 0.93 | 0.89 | 0.94 | 0.945 | 0.93 | 0.915 |
| (90,40) | $(40,0^{39})$ | | | | | | |
| AE | | 0.7309 | 0.5086 | 0.5571 | 0.3832 | 0.6208 | 0.4587 |
| Bias | | 0.1595 | 0.1086 | -0.0144 | -0.0167 | 0.0493 | 0.0587 |
| MSE | | 0.0668 | 0.0424 | 0.0022 | 0.0027 | 0.0089 | 0.0126 |
| CI | | (0.5400,0.8170) | (0.3414,0.6989) | (0.5279,0.5783) | (0.3493,0.4082) | (0.5240,0.7145) | (0.3450,0.5729) |
| AL | | 0.2770 | 0.3574 | 0.1505 | 0.2079 | 0.0503 | 0.0589 |
| CP | | 0.925 | 0.91 | 0.955 | 0.955 | 0.93 | 0.925 |
| AE | $(2^{20},1^{10},0^{10})$ | 0.6809 | 0.4522 | 0.6362 | 0.4598 | 0.6183 | 0.4789 |
| Bias | | 0.1094 | 0.0522 | 0.0648 | 0.0598 | 0.0468 | 0.0789 |
| MSE | | 0.0539 | 0.0234 | 0.0115 | 0.0158 | 0.0116 | 0.0161 |
| CI | | (0.4578,0.6991) | (0.2710,0.5441) | (0.2285,0.4721) | (0.5274,0.7409) | (0.3503,0.6116) | (0.3354,0.6028) |
| AL | | 0.2413 | 0.2731 | 0.1234 | 0.1612 | 0.1589 | 0.1973 |
| CP | | 0.91 | 0.92 | 0.95 | 0.94 | 0.93 | 0.935 |
| AE | $(1^{40})$ | 0.7210 | 0.5043 | 0.6156 | 0.4543 | 0.7160 | 0.5792 |
| Bias | | 0.1496 | 0.1043 | 0.0442 | 0.0543 | 0.1446 | 0.1792 |
| MSE | | 0.0654 | 0.0407 | 0.0077 | 0.0101 | 0.0111 | 0.0118 |
| CI | | (0.5654,0.8275) | (0.3393,0.6841) | (0.5191,0.6973) | (0.3416,0.5519) | (0.5463,0.6668) | (0.3701,0.5263) |
| AL | | 0.2621 | 0.3448 | 0.1782 | 0.2103 | 0.1205 | 0.1562 |
| CP | | 0.925 | 0.91 | 0.94 | 0.935 | 0.925 | 0.905 |

We generated 10,000 MCMC samples and then discard the first 1000 random values. Table 1 and 2 summarizes the ML estimators and the Bayes estimators for the parameters $\theta$,$\beta$ and S(t)) via the censored sample. Furthermore, from this table, it seems that the Bayes estimates under the non-informative prior and the ML Estimator were close to each other. The approximate 95% confidence intervals were computed together with the corresponding length for each interval, as reported below in Table 1 and 2. From these tables, it was discovered that the average length of the confidence interval and the credible interval decreased as n and m increased. The coverage probabilities of the confidence intervals based on the likelihood are close to the nominal level of 0.95 for $\theta$ and $\beta$, and S (t =0.5, 1) as n grew larger, but failed to reach the desired level for small values of n. On the other hand, the coverage probabilities of the credible intervals approached the nominal level of 0.95 for $\theta$ and $\beta$ and S (t =0.5, 1) in most cases.

## 6. Real Data Analysis

In this section, we consider a real life data to demonstrate the proposed method and verify how our estimates work in practice. The dataset was initially considered by Chhikara and Folks (1977). It represents the 46 repair times (in hours) for an airborne communication transceiver. The ordered dataset is presented below:

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0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

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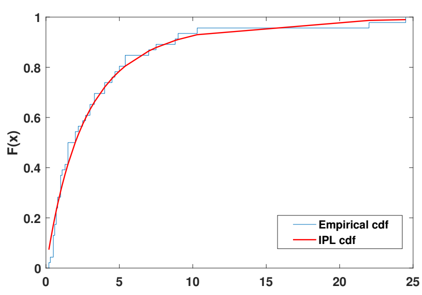


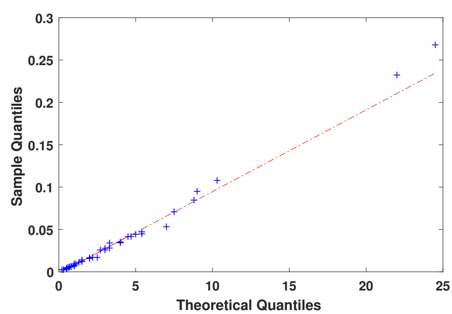


**Figure 4:** (a) ECDF plot for the dataset I (b) Q-Q plot for the dataset I

**Table 3:** Adaptive Progressive Type-II censored sample for n=46, m=20

| | |
|---|---|
| T=7.5, j=20 | $0.3, 0.5, 0.5, 0.5, 0.5, 0.7, 0.7, 1.0, 1.0, 1.5, 1.5, 2.0, 2.2, 2.5, 2.7,$ $3.3, 4.5, 4.7, 5.4, 7.0$ |
| T=2.0, j=11 | $0.2, 0.3, 0.5, 0.5, 0.6, 0.7, 1.0, 1.0, 1.0, 1.3, 1.5, 2.0, 2.2, 2.7, 4.0,$ $4.0, 4.75.4, 7.5, 22.0.$ |

In this illustration, the value of the Kolmogorov-Smirnov (K-S) distance and its corresponding p-value for the dataset are 0.1272 and 0.4462 respectively. It indicates that the dataset fits well through this distribution. This can further be seen through the visualization of the empirical Cumulative Distribution Function (ECDF) plot, the quantile-quantile (Q-Q) plot, as shown in Figure 4. The ML estimators for the unknown quantities are computed for the complete sample (uncensored), i.e. n=m, ($\theta$=0.1082 and $\beta$=3.5494) the dataset was used to simulate an adaptive progressive type-II censored sample with m = 20 and with two distinct values of ideal total test time T (2.0,7.5), as displayed in Table 3. For clarity $R = (5, 0^5)$ is given as a short form of $R = (5, 0, 0, 0, 0, 0)$. Thus, the observed adaptive progressive type-II censored samples are shown below in Table 3, for two different values of T and two distinct values of J. If J = 11 means that only 11 observed failures were observed before time T = 2.0 and J = 20 means that all the observed failure times were observed before time T = 7.5, then this implies that the experiment ended before time T. Table 4 and 5 represents the average estimates, CI and AL based on dataset I for the different values of T and R.

**Table 4:** AE, CI, and AL of $\theta$, $\beta$ and S(t), t=0.5,1 Parameters Based on Real dataset I for n=46, m=20, T=2.0, R=(20,$0^{19}$)

| **MLE** | | | | **Bayesian** | | | |
|---|---|---|---|---|---|---|---|
| $\theta$ | $\beta$ | **S(0.5)** | **S(1)** | $\theta$ | $\beta$ | **S(0.5)** | **S(1)** |
| 0.5277 | 0.3698 | 0.9170 | 0.8549 | 0.9188 | 0.9241 | 0.4620 | 0.3765 |
| (-0.1641,1.2197) | (0.0620,0.6776) | (0.7639,1.0700) | (0.6097,1.1000) | (0.0921,1.5362) | (0.1335,1.7083) | (0.1652,0.7379) | (0.0970,0.6672) |
| 1.3838 | 0.6156 | 0.3061 | 0.4903 | 1.4441 | 1.5747 | 0.5726 | 0.5701 |

**Table 5:** AE, CI, and AL of $\theta$, $\beta$ and S(t), t=0.5, 1 Parameters Based on Real dataset I for n=46, m=20, T=7.5, R=(10,$0^{18}$,10)

| **MLE** | | | | **Bayesian** | | | |
|---|---|---|---|---|---|---|---|
| $\theta$ | $\beta$ | **S(0.5)** | **S(1)** | $\theta$ | $\beta$ | **S(0.5)** | **S(1)** |
| 0.2651 | 1.0303 | 0.8796 | 0.7847 | 1.2675 | 0.9710 | 0.3795 | 0.2685 |
| (-0.2903,0.8206) | (-0.5284,1.2289) | (0.4890,1.2701) | (0.1478,1.4216) | (0.1240.1.7566) | (0.2088,1.5464) | (0.2746,0.6518) | (0.1441,0.5712) |
| 1.1103 | 1.7573 | 0.0.7811 | 1.2738 | 1.6325 | 1.3375 | 0.3772 | 0.4271 |

**Dataset 2**: The data represents the breakdown time of an insulating fluid between electrodes at a voltage of 34 kV studied by Nelson (1982). The data are recorded as follows:

---

0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16,4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89.

---

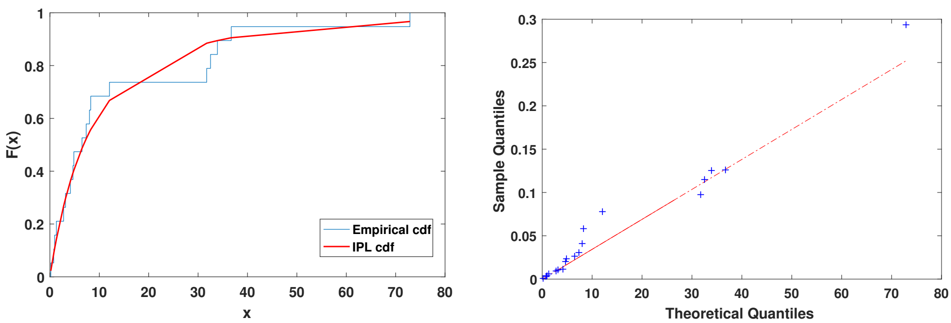


**Figure 5:** (a) ECDF plot for the dataset II (b) Q-Q plot for the dataset II

**Table 6:** Adaptive Progressive Type-II censored Sample for n=19 and m=10

| | |
|---|---|
| T=7, j=5 | 0.19, 0.96, 4.15, 4.85, 6.50, 8.01, 31.75, 32.52, 33.91, 36.71 |
| T=37, j=10 | 0.19, 1.31, 2.78, 3.16, 4.15, 4.85, 6.50, 32.52, 33.91, 36.71. |

In this illustration, the Kolmogorov-Smirnov (K-S) distance and its corresponding p-value for the dataset are 0.1479 and 0.7467 respectively. It indicates that the dataset fits well through this distribution. This can further be seen through the visualization of the empirical Cumulative Distribution Function (ECDF) plot, the quantile-quantile (Q-Q) plot as shown in Figure 5. The ML estimators for the complete sample (uncensored), i.e. n=m, ($\theta$=0.0597 and $\beta$=2.0323) the dataset was used to simulate an adaptive progressive type-II censored sample, as displayed in Table 6 with m = 10 and with two distinct values of ideal total test time T (7,37). Thus, the observed adaptive progressive type-II censored samples are shown below in Table 6, for two different numbers of T and two distinct numbers of J. If J =5 means that only 5 observed failures were observed before time T = 7 and J = 10 means that all the observed failure times were observed before time T = 37, then this implies that the experiment ended before time T. Table 7 and 8 presents the values of AEs, CI and AL based on dataset II for different values of T and R.

**Table 7:** AE, CI, and AL of $\theta$, $\beta$ and S(t), t=0.5, 1 Parameters Based on Real dataset II for n=19, m=10, T=37, R=(5,$0^8$,5)

| MLE | | | | Bayesian | | | |
|---|---|---|---|---|---|---|---|
| $\theta$ | $\beta$ | **S(0.5)** | **S(1)** | $\theta$ | $\beta$ | **S(0.5)** | **S(1)** |
| 0.1331 | 0.5242 | 0.9667 | 0.9365 | 1.1017 | 1.0280 | 0.3675 | 0.2657 |
| (-0.2363,0.5027) | (-0.3422,1.3906) | (0.8274,1.1061) | (0.6795,1.1935) | (0.1685.1.5000) | (0.3529,1.5228) | (0.2603,0.5294) | (0.1824,0.4358) |
| 1.3314 | 1.1698 | 0.2690 | 0.2534 | 0.5333 | 1.5530 | 0.4571 | 0.8790 |

**Table 8:** AE, CI, and AL of $\theta$, $\beta$ and S(t), t=0.5, 1 Parameters Based on Real dataset II for n=19, m=10, T=7, R=(5,$0^9$)

| MLE | | | | Bayesian | | | |
|---|---|---|---|---|---|---|---|
| $\theta$ | $\beta$ | **S(0.5)** | **S(1)** | $\theta$ | $\beta$ | **S(0.5)** | **S(1)** |
| 0.4370 | 1.0630 | 0.9772 | 0.9554 | 1.2704 | 1.0935 | 0.3067 | 0.1783 |
| (-0.2229,0.3103) | (0.2969,1.2561) | (0.7486,1.2058) | (0.5160,1.3950) | (0.9188.1.5265) | (0.9053,1.7482) | (0.9103,0.3543) | (0.1115,0.2101) |
| 0.5330 | 1.5530 | 0.4571 | 0.8790 | 0.6077 | 0.5428 | 0.1641 | 0.0985 |

## 7. Conclusion

In this study, the likelihood and Bayesian approaches were utilized to estimate the parameters of the Lomax distribution and survival function, under an adaptive progressive type-II censored data. However, closed-form solutions for the ML estimators of the parameters and survival function were unavailable, which led to the use of the Newton-Raphson numerical method for computation. Moreover, the study constructed asymptotic confidence intervals for $\theta$ and $\beta$, and an approximate confidence interval for the reliability function was obtained through the Delta method. The Bayesian approach used in the study employed both informative prior and non-informative prior. However, the Bayes estimates under the squared error loss function could not be derived analytically. As a result, the Metropolis-Hastings algorithm was utilized to generate 10,000 samples for estimation of the two unknown parameters, and credible intervals were computed for these quantities, as well as for the survival function. Furthermore, a simulation study was conducted to investigate the proposed methods for various sample sizes n, effective sample sizes m, and three different progressive censoring schemes, replicated 2000 times. The study also evaluated the proposed methods based on a real-life example. The estimators were observed to have small biases in all situations, indicating approximate unbiasedness. The average length of the estimators decreases with increase in the value of m and n. The MSEs of the estimators decreases with increase in the sample size. Overall, the study suggests that the Bayesian inference approach performs better than the classical approach. In the future endeavours, one could explore these estimation techniques in the presence of explanatory variables and develop more efficient computational algorithms to handle high-dimensional data and complex models. Further studies might also investigate the application of these generalized Lomax models in various domains, such as finance and biomedical sciences, to validate their practical utility.

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